STRONGLY BOUNDED LOCALLY INDICABLE GROUPS

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ABSTRACT. We give the construction of some locally indicable groups which are strongly bounded (every abstract action on a metric space has bounded orbits).

1. INTRODUCTION

A group G is *strongly bounded* if whenever G acts abstractly by isometries on a metric space every orbit is bounded. Infinite strongly bounded groups are necessarily uncountable. Examples of such groups include the full permutation group on a set [2] and powers of finite perfect groups [3]. Torsion-free examples were provided by Droste and Holland: the automorphism group of a doubly transitive total order [4]. For example $\mathrm{Aut}(\mathbb{Q},<)$ is strongly bounded. We construct strongly bounded groups whose properties are even stricter than those of $\mathrm{Aut}(\mathbb{Q},<)$. We remind the reader of some definitions before stating the main theorem.

We say that a total order $<$ on a group G is a *left-order* (respectively *right-order*) provided for all $g,h,k \in G$ we have that $g<h$ implies $kg<kh$ (resp. $gk<hk$). We say G is *left-orderable* provided there exists a left order on G. Left-orderable groups are torsion-free. A group order is a *bi-order* if it is both a left- and right-order and a group is *bi-orderable* provided such an order exists.

A group is *locally indicable* if every nontrivial finitely generated subgroup has a nontrivial map to the group $\mathbb{Z}$. Bi-orderable implies locally indicable implies left-orderable and neither of these implications is reversible [6]. The group $\mathrm{Aut}(\mathbb{Q},<)$ is left orderable but not locally indicable since every countable left-orderable group embeds as a subgroup (see proof of [6, Proposition 2.1]), and there exist countable left-orderable groups which are not locally indicable [1].

Using embedding theorems for HNN extensions we prove the following result, which gives the existence of strongly bounded groups with the strictest torsion-free local properties thus far in the literature.

Theorem 1.1. If K is locally indicable then there exists a simple, locally indicable, strongly bounded group $G \geq K$ with $|G| = |K|^{\aleph_0}$.

In particular there exist strongly bounded locally indicable groups of arbitrarily large cardinality, since a free group F of cardinality κ is locally indicable and the theorem gives a strongly bounded locally indicable group G of cardinality $\kappa^{\aleph_0}$.

It is not clear how to construct an infinite strongly bounded bi-orderable group but it seems likely that one exists.

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2. Proof

We remind the reader of some basic facts about locally indicable groups. The class of locally indicable groups includes all free groups, since nontrivial finitely generated subgroups of free groups are free of rank at least 1 and therefore indicable. Subgroups of locally indicable groups are obviously locally indicable. Moreover the class is closed under extensions: if $1 \to N \to G \to Q \to 1$ is a short exact sequence with N and Q locally indicable then so is G, since a finitely generated nontrivial subgroup of G will either lie inside of N or will map nontrivially to Q.

The class is also closed under taking free products. If A and B are locally indicable then so is $A \times B$ and the standard short exact sequence

$$1 \to F \to A * B \to A \times B \to 1$$

with F being a free subgroup of $A * B$, demonstrates that $A * B$ is locally indicable. By induction this class is closed under free products $*_{i\in I} G_i$ with I finite, and when I is infinite a finitely generated subgroup will lie inside of some $*_{i\in I'} G_i$ with $I' \subseteq I$ finite, so we indeed have closure under arbitrary free products.

We will make use of two results of Karrass and Solitar (see [5, Theorem 2] and [5, Theorem 6], respectively). The setup of these results is the following: Let J be a group and $\phi_i : A_i \to B_i$ be a collection of isomorphisms between subgroups $A_i, B_i \subseteq J$. Let L be the HNN extension $J *_{t_i A_i t_i^{-1} = \phi_i(B_i)}$.

Proposition 2.1. If J is locally indicable and each of the A_i is cyclic then L is locally indicable.

Proposition 2.2. If $H \leq L$ is a subgroup which has trivial intersection with all conjugates of A_i and B_i in L then H is the free product of a free group and the intersections of H with certain conjugates of J in L.

Construction 2.3. Suppose that $M \leq J$ are nontrivial torsion-free groups and that $\sigma : \mathbb{Z} \to M \setminus \{1\}$ is a function. Take L_0 to be the HNN extension of J given by $L_0 = J *_{t_z \langle \sigma(z) \rangle t_z^{-1} = \langle \sigma(z+1) \rangle_{z\in\mathbb{Z}}}$. Now the free group $F(\{t_z\}_{z\in\mathbb{Z}})$ is a retract subgroup of L_0 and we let ϕ be the automorphism on $F(\{t_z\}_{z\in\mathbb{Z}})$ for which $\phi(t_z) = t_{z+1}$. Let $E(M, J, \sigma)$ denote the HNN extension $L_0 *_{t\langle t_z \rangle t^{-1} = \langle t_{z+1} \rangle}$. This group $E(M, J, \sigma)$ will also be torsion-free by the standard theorems regarding HNN extensions, and also J naturally embeds as a subgroup of $E(M, J, \sigma)$.

Given a torsion-free group J we let $\{\sigma_\alpha\}_{\alpha < |J|^{\aleph_0}}$ be a well ordering of the functions $\sigma : \mathbb{Z} \to J \setminus \{1\}$. We define an increasing sequence $\{J_\alpha\}_{\alpha < |J|^{\aleph_0}}$ of torsion-free nesting groups. Let $J_0 = J$. If J_α has been defined for all $\alpha < \beta < |J|^{\aleph_0}$ and $\beta = \alpha + 1$ then let $J_\beta = E(J, J_\alpha, \sigma_\alpha)$. If β is a limit ordinal then let $J_\beta = \bigcup_{\alpha<\beta} J_\alpha$. Let $E(J)$ denote the union $\bigcup_{\alpha<|J|^{\aleph_0}} J_\alpha$. The construction of the group $E(J) \geq J$ formally depended, of course, on the well ordering of the σ. The order in which we took these HNN extensions actually does not make any difference up to the isomorphism class of $E(J)$, so the well ordering does not appear in the notation.

Lemma 2.4. The group $E(J)$ is locally indicable provided J is. If $|J| = |J|^{\aleph_0}$ then $|E(J)| = |J|$.

Proof. Since local indicability is preserved under infinite increasing unions, it suffices to show that if J is a locally indicable group, $M \leq J$, and $\sigma : \mathbb{Z} \to M \setminus \{1\}$ then $E(M, J, \sigma)$ is also locally indicable (by induction). To see that $E(M, J, \sigma)$ is locally indicable, we first notice that the extension L_0 defined in Construction 2.3 is locally indicable by Proposition 2.1. Next, we let $r : L_0 \to F(\{t_z\}_{z\in\mathbb{Z}})$ be the natural retraction. This extends to a retraction $r' : E(M, J, \sigma) \to F(\{t_z\}_{z\in\mathbb{Z}})*_{t\langle t_z\rangle t^{-1}=\langle t_{z+1}\rangle}$ by mapping $t \mapsto t$ and $g \mapsto r(g)$ for $g \in L_0$. The group $F(\{t_z\}_{z\in\mathbb{Z}})*_{t\langle t_z\rangle t^{-1}=\langle t_{z+1}\rangle}$ is a split extension

$$1 \to F(\{t_z\}_{z\in\mathbb{Z}}) \to F(\{t_z\}_{z\in\mathbb{Z}})*_{t\langle t_z\rangle t^{-1}=\langle t_{z+1}\rangle} \to \langle t\rangle \to 1$$

and so $F(\{t_z\}_{z\in\mathbb{Z}})*_{t\langle t_z\rangle t^{-1}=\langle t_{z+1}\rangle}$ is locally indicable as an extension of two locally indicable groups.

For the kernel $\ker(r') \le E(M, J, \sigma)$ it is clear that $\ker(r') \cap F(\{t_z\}_{z\in\mathbb{Z}})*_{t\langle t_z\rangle t^{-1}=\langle t_{z+1}\rangle}$ is trivial (since r' is a retraction). Thus more particularly $\ker(r') \cap F(\{t_z\}_{z\in\mathbb{Z}})$ is trivial. Since $\ker(r')$ is normal in $E(M, J, \sigma)$ we know that $\ker(r')$ has trivial intersection with all conjugates in $E(M, J, \sigma)$ of the subgroup $F(\{t_z\}_{z\in\mathbb{Z}})$. By Proposition 2.2 we have that $\ker(r')$ is a free product of a free group and groups which are isomorphic to subgroups of L_0. Thus $\ker(r')$ is locally indicable as a free product of locally indicable groups. Now $E(M, J, \sigma)$ is locally indicable as an extension

$$1 \to \ker(r') \to E(M, J, \sigma) \to F(\{t_z\}_{z\in\mathbb{Z}})*_{t\langle t_z\rangle t^{-1}=\langle t_{z+1}\rangle} \to 1$$

of locally indicable groups.

Suppose that $|J| = |J|^{\aleph_0}$. We have been assuming that J is nontrivial and so $|J|$ is uncountable. We see by induction that $|J| \le |J_\alpha| \le |J|^{\aleph_0}|J|^{\aleph_0} = |J|^{\aleph_0}$ for all $\alpha < |J|^{\aleph_0}$, and so $|E(J)| = |J|^{\aleph_0}$. □

Let ω denote the set of natural numbers. We use a necessary and sufficient criterion for strong boundedness given by de Cornulier (see [3, Proposition 2.7]).

Proposition 2.5. A group G is strongly bounded if and only for every function $\Lambda : G \to \omega$ such that for all $g, h \in G$ we have

- $\Lambda(1) \le 1$;
- $\Lambda(g) \le \Lambda(g^{-1}) + 1$; and
- $\Lambda(gh) \le \max(\Lambda(g), \Lambda(h)) + 1$

there exists some bound $P \in \omega$ for which $\Lambda(g) \le P$ for all $g \in G$.

Proof of Theorem 1.1. Let K be a locally indicable group. If K is the trivial group then we let $G = K$ and we are done. Suppose K is nontrivial. We can assume without loss of generality that $|K| = |K|^{\aleph_0}$ by replacing K with the free product of K with the free group of rank $|K|^{\aleph_0}$. We define G by an increasing nesting sequence $\{K_\alpha\}_{\alpha<\aleph_1}$ of supergroups of K. Let $K_0 = K$. If K_α has been defined for all $\alpha < \beta < \aleph_1$ and β is a limit ordinal then let $K_\beta = \bigcup_{\alpha<\beta} K_\alpha$. If $\beta = \alpha + 1$ then let $K_\beta = E(K_\alpha)$. Let $G = \bigcup_{\alpha<\aleph_1} K_\alpha$. Notice that each K_α has cardinality $|K|^{\aleph_0}$.

We have $|K| = |K|^{\aleph_0} \le |G| \le \aleph_1 \cdot |K|^{\aleph_0}$, and so G has the correct cardinality. The group G is also locally indicable as an increasing union of locally indicable groups K_α by Lemma 2.4. That G is simple follows from the fact that any two nontrivial elements are conjugate. More particularly, given $g, h \in G \setminus \{1\}$ we select $\alpha < \aleph_1$ for which $g, h \in K_\alpha$, let $\sigma : \mathbb{Z} \to K_\alpha \setminus \{1\}$ be given by

$$\sigma(n) = \begin{cases} g \text{ if } n \ge 0 \\ h \text{ if } n < 0 \end{cases}.$$

When σ appears in the definition of $K_{\alpha+1} = E(K_\alpha)$ we produce an element t_{-1} for which $t_{-1} h t_{-1}^{-1} = g$.

Finally we show that G is strongly bounded. The check will follow somewhat along the lines of the proof of [3, Theorem 3.1]. Suppose to the contrary, so that there exists an unbounded function $\Lambda : G \to \omega$ as in Proposition 2.5. Select a sequence $\{g_n\}_{n\in\omega}$ of nontrivial elements in G for which $\Lambda(g_n) \geq n^2$. By how G is constructed we may select $\alpha < \aleph_1$ for which $\{g_n\}_{n\in\omega} \subseteq K_\alpha$. Let $\sigma : \mathbb{Z} \to K_\alpha \setminus \{1\}$ be given by

$$\sigma(z) = \begin{cases} g_n \text{ if } z = n \geq 0 \\ g_0 \text{ if } z < 0 \end{cases}.$$

In $K_{\alpha+1} = E(K_\alpha)$ there exists a collection of elements $\{t_z\}_{z\in\mathbb{Z}}$ and t for which for all $z \in \mathbb{Z}$ we have $t_z\sigma(z)t_z^{-1} = \sigma(z+1)$ and $tt_zt^{-1} = t_{z+1}$. Select $M \in \omega$ large enough that $\Lambda(1), \Lambda(g_0), \Lambda(t_0), \Lambda(t_0^{-1}), \Lambda(t), \Lambda(t^{-1}) \leq M$. We clearly have $\Lambda(t^n), \Lambda(t^{-n}) \leq M + n$ for all $n \in \omega$. Thus

$$\Lambda(t_n^{\pm 1}) = \Lambda(t^n t_0^{\pm 1} t^{-n}) \leq M + n + 2$$

for all $n \in \omega$, from which we have by induction for $n \geq 1$ that

$$\Lambda(g_n) = \Lambda(t_{n-1} g_{n-1} t_{n-1}^{-1}) \leq M + 2n + 1$$

and thus $n^2 \leq \Lambda(g_n) \leq M + 2n + 1$ for all $n \in \omega$, a contradiction. □

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