BI-ORDERS DO NOT ARISE FROM TOTAL ORDERS

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Abstract. We present a Zermelo-Fraenkel consistency result regarding bi-orderability of groups. A classical consequence of the ultrafilter lemma is that a group is bi-orderable if and only if it is locally bi-orderable. We show that there exists a model of ZF plus dependent choice in which there is a group which is locally free (ergo locally bi-orderable) and not bi-orderable, and the group can be given a total order. The model also includes a torsion-free abelian group which is not bi-orderable but can be given a total order.

1. Introduction

The goal of this note is to explore the set theoretic strength of bi-orderability in the setting of Zermelo-Fraenkel set theory. Let **ZF** denote Zermelo-Fraenkel set theory minus **AC**, the axiom of choice. Recall that a *total order* on a set X is a binary relation $<$ for which exactly one of $x < y$ or $y < x$ holds for distinct $x, y \in X$, $x < x$ is false for all $x \in X$, and $x < y$ and $y < z$ imply $x < z$.

If G is a group we say that a total order $<$ on G is a *left-order* (respectively *right-order*) provided for all $g, h, k \in G$ we have that $g < h$ implies $kg < kh$ (resp. $gk < hk$). We say G is *left-orderable* provided there exists a left order on G. One could similarly define *right-orderable* but since a left-order explicitly defines a right-order and vice-versa, questions of left- or right-orderability of a group are equivalent. Left-orderable groups are torsion-free. A group order is a *bi-order* if it is both a left- and right-order and a group is *bi-orderable* provided such an order exists.

The ultrafilter lemma (every filter on a set extends to an ultrafilter) implies the classically known local-to-global bi-orderability result (see [4, Proposition 1.4]):

A group G is bi-orderable if and only if every finitely generated subgroup is bi-orderable.

In a nice setting one can have explicit bi-orders without having recourse to this local-to-global theorem. Given a total order on a set X one immediately obtains a bi-order on the free abelian group $F_{ab}(X)$ generated by X by considering the lexicographic order, and a bi-order on a free abelian group restricts to a total order on the free set of generators. Importantly the assertion that every set can be given a total order cannot be proved from **ZF**, so a total order on an arbitrary set X does not exist a priori. Thus in **ZF** a free abelian group is bi-orderable if and only if a free set of generators can be given a total order. By a more elaborate argument, in **ZF** a free group is bi-orderable if and only if a free generating set has a total

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order [2]. It seems natural to ask whether in **ZF** a total order on a locally free, or a torsion-free abelian, group implies bi-orderbility (by a total order on a group we mean, of course, a total order on the group's underlying set). We show that this is not even the case in the presence of dependent choices.

Recall that the *principle of dependent choices* is the assertion that if R is a binary relation on a nonempty set X for which $(\forall x \in X)(\exists y \in X)[xRy]$ then there exists a sequence $\{x_n\}_{n\in\omega}$ for which $x_n R x_{n+1}$. This principle, which is a consequence of the axiom of choice, implies many of the standard results in analysis and also implies the axiom of countable choices.

Theorem 1.1. If **ZF** is consistent then there exists a model of **ZF** in which the following hold:

(1) There exists a group $\mathcal{G}$ which is locally free and can be given a total order, but $\mathcal{G}$ is not bi-orderable.
(2) There exists a torsion-free abelian group $\mathcal{A}$ which can be given a total order, but $\mathcal{A}$ is not bi-orderable.
(3) The principle of dependent choices.

The overall strategy in this independence proof is to work in a permutation model of set theory, constructing the claimed groups via presentations, and using the permutations of the model to eliminate any possibility of a bi-order.

We leave some remaining questions regarding bi-orderability. We have mentioned that the local-to global bi-orderability theorem, which we'll denote **LG**, follows from the ultrafilter lemma. Also, **LG** implies the ordering principle (every set can be totally ordered) by considering the free abelian group on a set, which is locally free abelian and therefore locally bi-orderable. Thus we ask:

Question 1.2. Is **LG** strictly weaker than the ultrafilter lemma?

Question 1.3. Is **LG** strictly stronger than the ordering principle?

Since the ultrafilter lemma is strictly stronger than the ordering principle [3], the answer to at least one of the two above questions is "yes".

2. The proof

We will work in a modification of the model of van Douwen (see [8] or [1, Model $\mathcal{N}2(LO)$]). We let $\mathcal{M}$ be a model of **ZFA** + **AC** with set A of atoms such that $|A| = \aleph_1$. Write A as a disjoint union $A = \bigcup_{\alpha<\aleph_1} A_\alpha$ with each A_α being countably infinite and endowed with a total order $<_\alpha$ which makes A_α order isomorphic to $\mathbb{Z}$. Let Γ be the set of bijections τ on A for which $\tau \upharpoonright A_\alpha \in \mathrm{Aut}(A_\alpha, <_\alpha)$ for all $\alpha < \aleph_1$. Let $\mathcal{F}$ be the normal filter on Γ given by the ideal of countable subsets of A. Let $\mathcal{N} \subseteq \mathcal{M}$ denote the permutation model of hereditarily $\mathcal{F}$-symmetric objects in $\mathcal{M}$. For each $B \subseteq A$ we let $\mathrm{fix}(B) = \{\tau \in \Gamma \mid (\forall a \in B)\tau(a) = a\}$ and for an object $x \in \mathcal{M}$ we let $\mathrm{stab}(x) = \{\tau \in \Gamma \mid \tau(x) = x\}$. For each $a \in A$ we let $s(a)$ denote the next largest element under $<_\alpha$ in A_α, where $a \in A_\alpha$.

That the model $\mathcal{N}$ satisfies the principle of dependent choices follows from the fact that the ideal defining the filter $\mathcal{F}$ is closed under taking countable unions (see [1, Note 144]). It is not difficult to see that the existence of the claimed groups in Theorem 1.1 is boundable in the sense of Pincus [5]. Thus our main result will follow from the transfer principle [6, Theorem 4] (or see [1, page 286]) provided that we can establish the existence of the claimed groups in the model $\mathcal{N}$.

Let $J = A \times \{0,1\}$ and $\mathbb{F}(J) = (W_J, \circ_J, ^{-1}, 1_{W_J})$ denote the free group on the set J, with W_J denoting the set of reduced words over the alphabet $J^{\pm 1}$,$\circ_J$ and $^{-1}$ denoting the group multiplication and group inversion operations, and 1_{W_J} denoting the trivial element. This group, which we have defined in $\mathcal{M}$, is clearly in $\mathcal{N}$ as well; moreover, $\mathrm{stab}(W_J) = \mathrm{stab}(\circ_J) = \mathrm{stab}(^{-1}) = \Gamma$. Notice that the subset $X_J = \{(s(a),0)(a,1)(s(a),0)^{-1}(s(a),1)\}_{a\in A} \subseteq W_J$ is also in $\mathcal{N}$ and also supported by $\varnothing \subseteq A$. Therefore the normal subgroup $N_J = \langle\langle X_J\rangle\rangle \trianglelefteq \mathbb{F}(J)$ is in $\mathcal{N}$ and supported by $\varnothing$, and the similar claims hold for the quotient $\mathcal{G} = \mathbb{F}(J)/N_J$. We emphasize that the identity element N_J of $\mathcal{G}$, which we'll denote $1_{\mathcal{G}}$, is supported by $\varnothing$.

$\mathcal{G}$ is locally free. For each $\alpha \in \omega$ let $J_\alpha = (\bigcup_{\beta\le\alpha} A_\beta) \times \{0,1\}$. Similarly define the free group $\mathbb{F}(J_\alpha)$ and notice that $\mathbb{F}(J_\alpha)$ is in $\mathcal{N}$ and the set of reduced words in $J_\alpha^{\pm1}$, the group multiplication operation and the inverse operation are supported by $\varnothing$. Let $r_\alpha : \mathbb{F}(J) \to \mathbb{F}(J_\alpha)$ denote the retraction map given by deleting all letters in $J^{\pm1} \setminus J_\alpha^{\pm1}$ and freely reducing, and notice that r_α is in $\mathcal{N}$ and supported by $\varnothing$. Letting $\mathcal{G}_\alpha = F(J_\alpha)N_J$ we see that $\mathcal{G}_\alpha$ is also in $\mathcal{N}$ and supported by $\varnothing$. Also, $r_\alpha(X_J) \subseteq X_J \cup \{1_{W_J}\}$ and so $r_\alpha(N_J) \subseteq N_J$. Then the retraction homomorphism $\mathcal{G} \to \mathcal{G}_n$ given by taking a coset K of N_J to $r_n(K)N_J$ is in $\mathcal{N}$ and is similarly invariant under Γ.

We will show that each $\mathcal{G}_\alpha$ is locally free and this is sufficient since any finitely generated subgroup of $\mathcal{G}$ includes into some $\mathcal{G}_\alpha$. Fix $\alpha \in \omega$. Let T_α denote the group

$$\mathbb{F}(J_\alpha)/\langle\langle\{(s(a),0)(a,1)(s(a),0)^{-1}(s(a),1)\}_{a\in\bigcup_{\beta\le\alpha} A_\beta}\rangle\rangle.$$

It is easy to see that T_α is in $\mathcal{N}$ and that the identity map on the generators induces an isomorphism with $\mathcal{G}_\alpha$ (and this isomorphism is also in $\mathcal{N}$). We establish that T_α is locally free.

By selecting $a_\beta \in A_\beta$ for each $\beta \le \alpha$ we have $\mathrm{fix}(\{a_\beta\}_{\beta\le\alpha}) = \mathrm{fix}(\bigcup_{\beta\le\alpha} A_\beta)$. Since the object T_α is hereditarily supported by $\mathrm{fix}(\{a_\beta\}_{\beta\le\alpha})$ and $\mathcal{M}$ is a model of **ZFA** + **AC**, we may use **AC** in arguing that T_α is locally free. It is clear that T_α is the free product of $|\alpha| + 1$ copies of the group H given by presentation.

$$\langle \{x_m\}_{m\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}} \mid \{y_n = x_{n+1}^{-1} y_{n+1}^{-1} x_{n+1}\}_{n\in\mathbb{Z}}\rangle \tag{1}$$

Since the class of locally free groups is closed under taking free products, we now need to show that H is locally free.

Lemma 2.1. The group H is locally free and all generators $\{x_m\}_{m\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$ are nontrivial elements in H.

Proof. Notice that for a fixed $N \in \mathbb{Z}$ the presentation defining H does not require the relators $\{y_n = x_{n+1}^{-1} y_{n+1}^{-1} x_{n+1}\}_{n<N}$ and the generators $\{y_n\}_{n<N}$ since the relators $\{y_n = x_{n+1}^{-1} y_{n+1}^{-1} x_{n+1}\}_{n<N}$ are only used in giving names to the elements $\{y_n\}_{n<N}$. This is because for any positive $k \in \omega$ we can write

$$y_{N-k} = x_{N-k+1}^{-1} x_{N-k+2}^{-1} \cdots x_N^{-1} y_N^{(-1)^k} x_N \cdots x_{N-k+2} x_{N-k+1}$$

In particular, for any fixed $N \in \mathbb{Z}$ we know that H is isomorphic to the group H_N with presentation

$$\langle \{x_m\}_{m\in\mathbb{Z}} \cup \{y_n\}_{n\ge N} \mid \{y_n = x_{n+1}^{-1} y_{n+1}^{-1} x_{n+1}\}_{n\ge N}\rangle \tag{2}$$

via the map ρ_N determined by

$$x_m \mapsto x_m \text{ for all } n \in \mathbb{Z}$$
$$y_n \mapsto y_n \text{ for } n \geq N$$
$$y_{N-k} \mapsto x_{N-k+1}^{-1} x_{N-k+2}^{-1} \cdots x_N^{-1} y_N^{(-1)^k} x_N \cdots x_{N-k+2} x_{N-k+1} \text{ for } k \geq 1$$

Consider the normal subgroup $K = \langle\langle \{x_n\}_{n>N} \rangle\rangle \trianglelefteq H_N$. The quotient H_N/K has presentation

$$\langle \{x_m\}_{m \leq N} \cup \{y_n\}_{n \geq N} \mid \{y_n = y_{n+1}^{-1}\}_{n \geq N} \rangle$$

and this group is simply the free group in the generators $\{x_m\}_{m\leq N} \cup \{y_N\}$. This implies that for each $N \in \mathbb{Z}$ the set $\{x_m\}_{m \leq N} \cup \{y_N\}$ freely generates a subgroup of H_N.

For any finite set of words $\{w_0, \ldots, w_r\}$ in the letters $\{x_m\}_{m\in\mathbb{Z}}^{\pm 1} \cup \{y_n\}_{n\in\mathbb{Z}}^{\pm 1}$ there exists some N for which each of the words $w_0, \ldots, w_r$ is written in the letters $\{x_m\}_{m\leq N}^{\pm 1} \cup \{y_n\}_{n \leq N}^{\pm 1}$. Then applying ρ_N to the group elements represented by $w_0, \ldots, w_r$ places this set within the subgroup $\langle \{x_m\}_{m \leq N} \cup \{y_N\} \rangle \leq H_N$, and since this subgroup is free, we have that H is locally free. The second claim follows immediately from our proof since we showed that for each $N \in \mathbb{Z}$ the set $\{x_m\}_{m \leq N} \cup \{y_N\}$ freely generates a subgroup of H. □

$\mathcal{G}$ can be given a total order. Towards producing a total order on $\mathcal{G}$ we produce, in $\mathcal{M}$, a normal form for $\mathcal{G}$. Since **AC** holds in $\mathcal{M}$ we shall freely use choices in this construction, and the fact that the normal form is also in $\mathcal{N}$ will become apparent. Recall that a word rewriting system on a free monoid $\mathrm{Mon}(X)$ on set X is a set of rules $\mathcal{R}$ whose inputs and outputs are words in the monoid (see [7, Section 1.7]). We define binary relation $\to_{\mathcal{R}}$ on $\mathrm{Mon}(X)$ by letting $w_0 \to_{\mathcal{R}} w_1$ if there exist $v_0, v_1, v_1', v_2 \in \mathrm{Mon}(X)$ with $w_0 \equiv v_0 v_1 v_2$ and $w_1 \equiv v_0 v_1' v_2$ and $(v_1, v_1') \in \mathcal{R}$. Let $\to_{\mathcal{R}}^*$ be the smallest transitive binary relation including $\to_{\mathcal{R}}$ and let $\leftrightarrow_{\mathcal{R}}^*$ denote the smallest equivalence class including $\to_{\mathcal{R}}^*$. Rewriting system $\mathcal{R}$ is *confluent* if whenever $w_0 \to_{\mathcal{R}}^* w_1$ and $w_0 \to_{\mathcal{R}}^* w_2$ there exists w_3 for which $w_1 \to_{\mathcal{R}}^* w_3$ and $w_2 \to_{\mathcal{R}}^* w_3$. It is *locally confluent* if whenever $w_0 \to_{\mathcal{R}} w_1$ and $w_0 \to_{\mathcal{R}} w_2$ there exists w_3 for which $w_1 \to_{\mathcal{R}}^* w_3$ and $w_2 \to_{\mathcal{R}}^* w_3$.

Rewriting system $\mathcal{R}$ is *terminating* if each sequence $w_0 \to_{\mathcal{R}} w_1 \to_{\mathcal{R}} w_2 \cdots \to_{\mathcal{R}} w_n$ must eventually stabilize. A word w is a *terminus* of $\mathcal{R}$ if $w \to_{\mathcal{R}} v$ implies $w \equiv v$. If $\mathcal{R}$ is terminating and locally confluent then it is confluent, and if $\mathcal{R}$ is terminating and confluent then each equivalence class in $\leftrightarrow_{\mathcal{R}}^*$ contains a unique terminus (see [7, Section 1.7]).

We let $\mathrm{Mon}(J^{\pm 1})$ denote the free monoid on the set $\{(a,0)\}_{a\in A} \cup \{(a,1)\}_{a\in A} \cup \{(a,0)^{-1}\}_{a\in A} \cup \{(a,1)^{-1}\}_{a\in A}$, and let e denote the empty word. Consider the rewriting system $\mathcal{R}$ under which for all $a \in A$ we have rules

(i) $(a,0)(a,0)^{-1} \mapsto e$
(ii) $(a,0)^{-1}(a,0) \mapsto e$
(iii) $(a,1)(a,1)^{-1} \mapsto e$
(iv) $(a,1)^{-1}(a,1) \mapsto e$
(v) $(s(a),0)(a,1) \mapsto (s(a),1)^{-1}(s(a),0)$
(vi) $(s(a),0)(a,1)^{-1} \mapsto (s(a),1)(s(a),0)$
(vii) $(s(a),0)^{-1}(s(a),1) \mapsto (a,1)^{-1}(s(a),0)^{-1}$
(viii) $(s(a),0)^{-1}(s(a),1)^{-1} \mapsto (a,1)(s(a),0)^{-1}$

The idea of this system is to both freely reduce and to move the $(a,0)^{\pm 1}$ letters to the right.

Lemma 2.2. The rewriting system $\mathcal{R}$ is locally confluent.

Proof. We'll argue in cases. It is easy to see that if rules are applied independently to non-overlapping subwords then the order of application makes no difference. More explicitly if we have a word $w \equiv u_0u_1$ and consider an application of a rule to u_0 to obtain $w \to_{\mathcal{R}} u_0'u_1$, and consider the application of a possibly different rule to u_1 to obtain $w \to_{\mathcal{R}} u_0u_1'$ then clearly $u_0'u_1 \to_{\mathcal{R}} u_0'u_1'$ and $u_0u_1' \to_{\mathcal{R}} u_0'u_1'$ and so there can be no obstruction to local confluence in this setting. Thus it will only be necessary to consider cases where rule applications are to overlapping subwords.

If $w \to_{\mathcal{R}} w_0$ and $w \to_{\mathcal{R}} w_1$ are each obtained by an application of a free reduction rule (i.e. each is obtained by one of (i) - (iv)) then by applying free reductions to each of w_0 and w_1 we obtain a unique freely reduced word w_2, so that $w_0 \to_{\mathcal{R}}^* w_2$ and $w_1 \to_{\mathcal{R}}^* w_2$. Next, if $w \to_{\mathcal{R}} w_0$ and $w \to_{\mathcal{R}} w_1$ and each of these was given by an application of possibly different rules among (v) - (viii) then either $w_0 \equiv w_1$ or these rules were applied on distinct non-overlapping subwords, and this latter case was considered above.

Next we suppose that $w \equiv v_0(s(a),0)(a,1)(a,1)^{-1}v_1$. By applying (iii) one has $w \to_{\mathcal{R}} v_0(s(a),0)v_1$. By instead applying (v) to w we see that

$$w \to_{\mathcal{R}} v_0(s(a),1)^{-1}(s(a),0)(a,1)^{-1}v_1$$

and by applying (vi) and then (iv) we see that

$$v_0(s(a),1)^{-1}(s(a),0)(a,1)^{-1}v_1 \to_{\mathcal{R}} v_0(s(a),1)^{-1}(s(a),1)(s(a),0)v_1 \to_{\mathcal{R}} v_0(s(a),0)v_1.$$

The cases where w is of form

$$w \equiv v_0(s(a),0)(a,1)^{-1}(a,1)v_1;$$
$$w \equiv v_0(s(a),0)^{-1}(s(a),1)(s(a),1)^{-1}v_1; \text{ or}$$
$$w \equiv v_0(s(a),0)^{-1}(s(a),1)^{-1}(s(a),1)v_1$$

are each handled similarly.

Suppose that we have a word $w \equiv v_0(s(a),0)^{-1}(s(a),0)(a,1)v_1$. If one applies (ii) then one has $w \to_{\mathcal{R}} v_0(a,1)v_1$. If we instead apply (v) then we get $w \to_{\mathcal{R}} v_0(s(a),0)^{-1}(s(a),1)^{-1}(s(a),0)v_1$, and applying rule (viii) and then (ii) we get

$$v_0(s(a),0)^{-1}(s(a),1)^{-1}(s(a),0)v_1 \to_{\mathcal{R}} v_0(a,1)(s(a),0)^{-1}(s(a),0)v_1 \to_{\mathcal{R}} v_0(a,1)v_1.$$

The check in case w is of form

$$w \equiv v_0(s(a),0)^{-1}(s(a),0)(a,1)^{-1}v_1;$$
$$w \equiv v_0(s(a),0)(s(a),0)^{-1}(s(a),1)v_1; \text{ or}$$
$$w \equiv v_0(s(a),0)(s(a),0)^{-1}(s(a),1)^{-1}v_1$$

is similar. Thus local confluence holds. □

We note also that the rewriting system is terminating. To see this, given a word w we consider the function

$$j(w) = \sum_{0 \le i < \mathrm{Len}(w), w(i) \in \{(a,0)^{\pm 1}\}_{a \in A}} |\{i < k < \mathrm{Len}(w) \mid w(k) \in \{(a',1)^{\pm 1}\}_{a' \in A}\}|$$

which counts the total number of times that a letter of form $(a',1)^{\pm 1}$ appears in the word somewhere to the right of a letter of form $(a,0)^{\pm 1}$. Each application of a rule will lower the value of the function $\mathrm{Len}(w) + j(w)$ (where $\mathrm{Len}(\cdot)$ denotes the length of the word) and so the fact that the system is terminating follows. Thus each equivalence class under $\leftrightarrow_{\mathcal{R}}^*$ contains a unique terminus.

All elements of the set R of words which are the terminus of a word in $\mathrm{Mon}(J^{\pm 1})$ under $\mathcal{R}$ are freely reduced. The set R is also obviously in $\mathcal{N}$ (notice that the rules are themselves invariant under the action of Γ) and supported by $\varnothing$. Furthermore it is straightforward to see that each element in R is a unique representative of an element in $\mathcal{G}$. We give an order $<^l$ to the letters in $J^{\pm 1}$ as follows:

$$(a,0)^{-1} <^l (a,0) <^l (a,1)^{-1} <^l (a,1) <^l (a',0)^{-1} <^l (a',0) <^l (a',1)^{-1} <^l (a',1)$$

where either $a, a' \in A_\alpha$ with $a <_\alpha a'$ or $a \in A_\alpha$ and $a' \in A_{\alpha'}$ with $\alpha < \alpha'$. Endow the elements of R with the shortlex order $<^o$: $w_0 <^o w_1$ if either $\mathrm{Len}(w_0) < \mathrm{Len}(w_1)$, or $\mathrm{Len}(w_0) = \mathrm{Len}(w_1)$ and for the least $0 \le i < \mathrm{Len}(w_0)$ at which $w_0(i) \ne w_1(i)$ we have $w_0(i) <^l w_1(i)$. It is clear that both $<^l$ and $<^o$ are in $\mathcal{N}$, and more particularly they are supported by $\varnothing$.

$\mathcal{G}$ is not bi-orderable. To see that $\mathcal{G}$ is not bi-orderable we suppose for contradiction that $<_\mathcal{G}$ is a bi-order on $\mathcal{G}$ in $\mathcal{N}$. Select countable $B \subseteq A$ for which $\mathrm{fix}(B) \le \mathrm{stab}(<_\mathcal{G})$. Select $\alpha < \aleph_1$ such that $A_\alpha \cap B = \varnothing$. Let $\tau \in \Gamma$ be given by

$$\tau(a) = \begin{cases} a \text{ if } a \notin A_\alpha \\ s(a) \text{ if } a \in A_\alpha \end{cases}$$

Let $a \in A_\alpha$ be given. By Lemma 2.1 we know that $(a,1)N_J$ is nontrivial. If $1_\mathcal{G} <_\mathcal{G} (a,1)N_J$ then $1_\mathcal{G} <_\mathcal{G} (s(a),0)(a,1)(s(a),0)^{-1}N_J = (s(a),1)^{-1}N_J$, from which we see that $(s(a),1)N_J <_\mathcal{G} 1_\mathcal{G}$, but on the other hand

$$1_\mathcal{G} = \tau(1_\mathcal{G}) <_\mathcal{G} \tau((a,1))N_J = (s(a),1)N_J$$

which is a contradiction. The proof in case $(a,1)N_J <_\mathcal{G} 1_\mathcal{G}$ is symmetric.

The group $\mathcal{A}$ and its properties. We construct the group $\mathcal{A}$ which is claimed to be in the model $\mathcal{N}$. In checking its various properties, we'll simply sketch over the aspects of the proofs which are nearly identical to those in the case of $\mathcal{G}$. We take $\mathbb{F}(A)$ to be the free group on the set A of atoms. Consider the subset $X_A = \{[a,a']\}_{a,a' \in A} \cup \{a(s(a))^2\}_{a \in A}$, where $[a,a']$ denotes the commutator $aa'a^{-1}(a')^{-1}$. This set is in $\mathcal{N}$ and supported by $\varnothing$, and similarly for the normal subgroup $N_A = \langle\langle X_A \rangle\rangle$ and relevant group operations and underlying set of $\mathcal{A} = \mathbb{F}(A)/N_A$. Letting $0_\mathcal{A}$ denote the identity element, we emphasize that $0_\mathcal{A}$ is supported by $\varnothing$. Let $B_\alpha = \bigcup_{\beta \le \alpha} A_\beta$ and $r_\alpha : \mathbb{F}(A) \to \mathbb{F}(B_\alpha)$ be the retraction. Let $\mathcal{A}_\alpha = \mathbb{F}(B_\alpha)N_A$. Since $r_\alpha(Y) \subseteq Y \cup \{1\}$ we have $r_n(N_A) \subseteq N_A$. Thus we have a retraction map $\mathcal{A} \to \mathcal{A}_\alpha$ given by $K \mapsto r_\alpha(K)N_A$ which is in $\mathcal{N}$ and supported by $\varnothing$.

Let L_α denote the group $\mathbb{F}(B_\alpha)/\langle\langle \{[a,a']\}_{a,a' \in B_\alpha} \cup \{a(s(a))^2\}_{a \in B_\alpha} \rangle\rangle$. Notice that $L_n \simeq \mathcal{A}_n$ via the identity map on the generators (and this isomorphism is in $\mathcal{N}$). Taking $a_\beta \in A_\beta$ for each $\beta \le \alpha$ we have again that $\mathrm{fix}(\{a_\beta\}_{\beta \le \alpha}) = \mathrm{fix}(B_\alpha)$. Thus we may utilize **AC**, which holds in $\mathcal{M}$, in analyzing L_α. It is easy to see that L_α is isomorphic to a direct sum of $|\alpha| + 1$ copies of the additive group $\mathbb{Z}[\frac{1}{2}]$. Thus $\mathcal{A}_\alpha$, and therefore all of $\mathcal{A}$, is torsion-free abelian and for each $a \in A$ we have aN_A nontrivial in $\mathcal{A}$.

A normal form on $\mathcal{A}$ is given by words of the form

$$a_0^{z_0} a_1^{z_1} \cdots a_m^{z_m}$$

where for each $0 \le i \le m$ we have $z_i \in \mathbb{Z} \setminus 2\mathbb{Z}$ and $a_i \in A_{j_i}$ with $j_0 < j_1 < \cdots < j_m$. The set of all such words is in $\mathcal{N}$ and supported by $\varnothing$. Order the letters $A^{\pm 1}$ by order $<^l$ given by

$$a^{-1} <^l a <^l (a')^{-1} <^l a'$$

where $a, a' \in A_\alpha$ for some $\alpha < \aleph_1$ and $a <_\alpha a'$, or $a \in A_\alpha$ and $a' \in A_{\alpha'}$ with $\alpha < \alpha'$. This order $<^l$ is invariant under Γ. Order $\mathcal{A}$ using shortlex on the normal form.

Now suppose that $<_\mathcal{A}$ is a bi-order on $\mathcal{A}$. Let $B \subseteq A$ be countable with $\mathrm{fix}(B) \leq \mathrm{stab}(<_\mathcal{A})$. Select $\alpha \in \omega$ such that $A_\alpha \cap B = \varnothing$. Let $\tau \in \Gamma$ be given by

$$\tau(a) = \begin{cases} a \text{ if } a \in A \setminus A_\alpha \\ s(a) \text{ if } a \in A_\alpha \end{cases} .$$

Let $a \in A_\alpha$. Suppose that $0_\mathcal{A} <_\mathcal{A} aN_A$. On one hand we have that $0_\mathcal{A} = \tau(0_\mathcal{A}) <_\mathcal{A} \tau(aN_A) = s(a)N_A$, but on the other hand we have $s(a)^2 N_A = a^{-1}N_A <_\mathcal{A} 0_\mathcal{A}$, a contradiction. The proof in case $aN_A <_\mathcal{A} 0_\mathcal{A}$ is symmetric.

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References

[1] P. Howard, J. E. Rubin, Consequences of the Axiom of Choice, Math. Surveys and Monographs 59, Amer. Math. Soc., Providence R. I., (1998).

[2] W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann. 111 (1935), 259-280.

[3] A. R. D. Mathias, *The order-extension principle*, Axiomatic set theory, Proceedings of Symposia in Pure Mathematics, XIII part 2, American Mathematical Society (1974), 179-184.

[4] A. Navas, *On the dynamics of (left) orderable groups*, Ann. Inst. Fourier 60 (2010), 1685-1740.

[5] D. Pincus, *Zermelo-Fraenkel consistency results by Fraenkel-Mostowski Methods*, J. Symb. Logic 37 (1972), 721-743.

[6] D. Pincus, *Adding dependent choice*, Ann. Math. Log. 11 (1977), 105-145.

[7] M. Sapir, Combinatorial algebra: Syntax and semantics, Springer Verlag (2014).

[8] E. van Douwen, *Horrors of topology without AC: A non-normal orderable space*, Proc. Amer. Math. Soc. 95 (1985), 101-105.

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