RESEARCH STATEMENT

SAMUEL M. CORSON

Abstract. A brief overview of my research interests, stating a handful of results and their context.

1. Introduction

There are strong connections among group theory, geometry, topology, and set theory. Such ties have been extensively studied and utilized over the years. For example, Gromov's celebrated polynomial growth theorem draws a precise connection between a geometric property of a group and an algebraic: a finitely generated group is virtually nilpotent if and only if it has polynomial growth ([32], [42]). As another example, the geometric notion of hyperbolicity [33] has influenced group theory for the last 30 years, with recent generalizing notions such as relative hyperbolicity [29], lacunary hyperbolicity [36], and acylindrical hyperbolicity [37]. Hyperbolicity implies a solution to the algorithmic word and conjugacy problems.

Model theoretic notions for groups also have connections to the geometry of a group. For example, any isometric action of an $\aleph_1$-existentially closed group on a metric space has bounded orbits [6]. Techniques of set theory can also show that a problem in group theory, such as the Whitehead Problem [39], cannot be resolved using the standard axioms of set theory.

I research group theory and topology, utilizing tools of geometry, combinatorics and set theory. I attack research both (**1**) from the side of establishing general rules and (**2**) from the side of constructing instructive examples. I'll describe the structure of this research statement.

In Section 2 are presented some automatic continuity theorems obtained with various coauthors (these fall under paradigm(**1**)). For example, any abstract group homomorphism from a Polish group to Thompson's group F or to a torsion-free word hyperbolic group will have open kernel [5]. Thompson's group may be replaced with many other groups, e.g. a braid group [22], and the conclusion still holds.

In Section 3 are theorems which fall under paradigm (**2**). For example, a result with Saharon Shelah which gives groups of many cardinalities whose every action on a metric space has bounded orbits (so-called *strongly bounded groups*) [24]. Also, a group which is "just barely not free" [14], a topological space which is "just barely connected" [18], groups which are "very much of cardinality κ" [20], and a couple of curious models of set theory without the axiom of choice [15], [17].

In Section 4 some results surrounding fundamental groups are given, which fall mostly under paradigm (**1**) (including, for example, decomposition theorems). A result in paradigm (**2**) is also mentioned: the fundamental group of the Griffiths double cone is isomorphic to that of the harmonic archipelago, affirming a conjecture of James Cannon and Greg Conner.

2. Automatic continuity

A staple of abelian group theory is the notion of slenderness (see [30]). An abelian group A is *slender* if for any homomorphism $\phi : \prod_{\mathbb{N}} \mathbb{Z} \to A$ there exists an $m \in \mathbb{N}$ such that $\phi = \phi \circ p_m$ (here $p_m : \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z}$ changes all coordinates after the first $m+1$ to zero). This class of abelian groups has been classified via subgroups [35], has an infinitary axiomatization [34], and has connections to measurable cardinals [26].

One may consider slenderness to be a strong form of automatic continuity. Endowing $\prod_{\mathbb{N}} \mathbb{Z}$ with the product topology with each constituent $\mathbb{Z}$ group being discrete, an abelian group A is slender precisely if every map $\phi : \prod_{\mathbb{N}} \mathbb{Z} \to A$ has open kernel. One analogously defines a group G to be *cm-slender* (respectively *lcH-slender*) if for every homomorphism $\phi : H \to G$ with H a completely metrizable (resp. locally compact Hausdorff) group we have $\ker(\phi)$ open in H. Slenderness also has an analogue in fundamental groups: G is *n-slender* if every homomorphism from the fundamental group of the infinite earring (we will denote this fundamental group EG) has open kernel [27].

Dudley showed that free groups and free abelian groups are cm- and lcH-slender [26]. Combining geometric and set theoretic arguments, Greg Conner and I show the following strengthening [5]:

Theorem 1. The following groups are n-, cm-, and lcH-slender:

(a) Countable torsion-free groups with finite k-antecedents for some $k \in \mathbb{N}$;
(b) Countable torsion-free groups with finite roots;
(c) Free groups;
(d) Free abelian groups;
(e) $\mathbb{Z}[\frac{1}{m}]$ for each $m \in \mathbb{N}$;
(f) Torsion-free word hyperbolic groups;
(g) Baumslag-Solitar groups;
(h) Thompson's group F.

We also show that the classes of n-, cm-, and lcH-slender groups are closed under graph products. In particular the right angled Artin groups are slender in each of these senses. In [12] I prove comparable results for groups whose every countable subgroup is free (such groups are the so called $\aleph_1$*-free groups*), and with Ilya Kazachkov we prove slenderness of braid groups [22]. With Oleg Bogopolski we give the following automatic continuity result for the class of acylindrically hyperbolic groups [3]:

Theorem 2. Let G be a group which acts coboundedly and acylindrically on a hyperbolic space S. Then for any abstract group homomorphism $\phi : H \to G$ where H is either

(a) a completely metrizable topological group;
(b) a locally countably compact Hausdorff topological group;
(c) the earring group EG.

there exists an open neighborhood V of identity 1_H such that $\phi(V)$ consists of elliptic elements.

One can conclude from this that if G is a torsion-free group of cardinality less than $2^{\aleph_0}$ which is relatively hyperbolic to a collection of cm-slender parabolic subgroups then G is itself cm-slender. Similar applications are given in [3] to fundamental groups of graphs of groups.

A succinct classification of lcH-slender groups was recently obtained with Olga Varghese [25]:

Theorem 3. A group is lcH-slender if and only if it is torsion-free and does not include a subgroup isomorphic to $\mathbb{Q}$ or to the p-adic integer group for any prime p.

A homomorphism from a completely metrizable topological group to a countable group whose kernel is not open is necessarily "nonconstructive". That is, the homomorphism is not even close to being defined using geometry and basic set theory. Shelah and I prove the following [23]:

Theorem 4. There exists a model of the Zermelo-Fraenkel axioms plus the axiom of dependent choices in which any homomorphism either from EG or a completely metrizable topological group to a group of cardinality $< 2^{\aleph_0}$ must have open kernel.

Once one knows that there exists at least one discontinuous homomorphism to a particular group it is reasonable to ask how many homomorphisms there are in total. After all, when many arbitrary choices are used in producing an object there is a sense that making different choices will yield a distinct object. This intuition is correct [13]:

Theorem 5. If G is a group such that $|G| < 2^{\aleph_0}$ then

$$|\operatorname{Hom}(\mathrm{EG}, G)| = \begin{cases} |G| & \text{if G is n-slender,} \\ 2^{2^{\aleph_0}} & \text{otherwise.} \end{cases}$$

3. Some Monsters

Another way in which set theory influences group theory is its use in the very construction of a group. One can construct spectacularly unusual groups using such machinery. One such construction is the following (the V = L means that the result is true in Gödel's constructible universe) [14]:

Theorem 6. (ZFC + V = L) Let κ be an uncountable regular cardinal that is not weakly compact there exists a group G of cardinality κ for which

(1) G is freely indecomposable;
(2) each subgroup of G of cardinality $< \kappa$ is free;
(3) the abelianization G/G' is free abelian of rank κ.

Such groups are unusual since they are evidently very close to behaving like a free group (by conditions (2) and (3)) but are very far from being free by condition (1). The cardinal in the hypotheses is allowed to be, for example, $\aleph_1$, $\aleph_2$, $\aleph_{\omega+1}$. The construction utilizes the set theoretic diamond principle $\diamondsuit$ and refines an old result obtained by Eklof and Mekler [28].

Another construction is of unusual groups which are "anti-geometric". A group is *strongly bounded* if every action of the group by isometries on any metric space has bounded orbits [6]. Obviously any finite group has this property, and tt turns out that any infinite group with this property must be uncountable. Bergman showed that for any set X the full symmetric group S_X on a set X has this property [2].

Many other classical groups also have this property. All previously known infinite examples of such groups have cardinality at least $2^{\aleph_0}$, and it is natural to ask whether there exist examples of such groups which are of cardinality exactly $\aleph_1$ [6]. With Saharon Shelah we construct such examples [24]:

Theorem 7. Every countable group embeds in a strongly bounded group of cardinality $\aleph_1$.

In fact a much stronger result is given (if κ is "almost any" uncountable cardinal then any group G with $|G| < \kappa$ embeds in a strongly bounded group of cardinality κ). The construction uses small cancellation theory over free products and strong Ramsey coloring theory. One can even have strongly bounded groups which are locally indicable [16].

An important recent construction answers some old questions in infinite group theory. A group G is *Jónsson* if $|H| < |G|$ for every proper subgroup H of G. A Jónsson group of cardinality $\aleph_1$ was constructed by Shelah in [40] and he also showed that assuming some extra set theory one can have Jónsson groups of arbitrarily large cardinality. It turns out that the extra set theory is not necessary. In [20] is proved the following (an old conjecture of László Babai [1] also follows from the work of this paper):

Theorem 8. There exist Jónsson groups of arbitrarily large cardinality.

Departing from group theory for a moment, I'll describe state a topological space which was recently obtained using extra set theory [18]. It would be tedious to give the various definitions, so I will simply state the result and briefly say why it is interesting.

Theorem 9. (ZFC + $\diamondsuit$) There exists a topological space (X, τ), having a basis $\mathcal{B}$, which is

(1) regular;
(2) separable;
(3) connected;
(4) of cardinality $\aleph_1$;

and

(5) for every nonempty $O \in \tau$ the subspace $X \setminus O$ is totally separated (hence totally disconnected);
(6) every nonempty $O \in \tau$ is uncountable;
(7) $|\mathcal{B}| = \aleph_1$;
(8) every countable cover of X by elements of $\mathcal{B}$ has a finite subcover; and
(9) every nesting decreasing countable sequence $\{B_n\}_{n\in\omega}$ of basis elements has $\bigcap_{n\in\omega} B_n \neq \emptyset$ (hence X is strongly Choquet).

Such a space is quite tenuously connected ((3) and (5)), but extremely topologically thick (9). The topology is reasonably fine (1) and narrow ((2) and (8)). One cannot have a space satisfying (1)-(9) which is metrizable or has a nonempty open subset with compact closure, but condition (8) is awfully close to countable compactness. Notice as well that usual examples satisfying condition (9) are totally disconnected (e.g. the Cantor set or a discrete space).

Along the topology theme, it was asked by Good, Tree and Watson [31] whether the famous theorem of Stone that every metric space is paracompact can be proved from the Zermelo-Fraenkel axioms (without the axiom of choice) plus the ultrafilter lemma. I show that this cannot be done in the paper [15], where a model of

ZF + "ultrafilter lemma" + "Stone's Theorem is false"

is given. The construction relies on analyzing the dynamics of a Fraïssé limit metric space.

Returning to group theory, there is a classical result, proved from the ultrafilter lemma, that a group has a left invariant order if and only if every finitely generated subgroup has a left order. Finitely generated free (abelian) groups are known to have such an order. A model is constructed in [17] of

ZF + "dependent choice"
+ "there exists a locally free group without a left order, but having an abstract total order"
+ "there exists a torsion-free abelian group without a left order, but having an abstract total order"

This highlights the fact that some fairly high-powered choice principle is needed in order to make a local-to-global claim regarding left-orderability.

4. Exotic spaces and descriptive set theory

One of the most basic connections between group theory and topology is the fundamental group. A celebrated result of Shelah is that the fundamental group of a compact, metrizable, path connected, locally path connected space (a *Peano continuum*) is either finitely generated or of cardinality $2^{\aleph_0}$ [41]. His proof used forcing techniques, and a simplified proof using descriptive set theory was later offered by Pawlikowski [38]. Greg Conner and I proved the analogous result for first homology: the first homology of a Peano continuum is either a finite direct sum of cyclic groups or of cardinality $2^{\aleph_0}$ [4]. I give the following higher dimensional analog in [7]:

Theorem 10. If X is a Peano continuum which is n-connected, locally n-connected then $\pi_{n+1}(X)$ is either a finite direct sum of cyclic groups or is of cardinality $2^{\aleph_0}$.

Also in [7] I give a consequence of Martin's Axiom plus the negation of the continuum hypothesis:

Theorem 11. (ZFC + MA + $\neg$ CH) If X is a path connected Borel set in a Polish space and $x \in X$ has essential loops of arbitrarily small diameter then $\pi_1(X)$ is of cardinality $2^{\aleph_0}$.

The proof technique of Theorem 11 does not generalize to a proof that takes place within the ZFC axioms. Numerous such dichotomies are presented in my paper [10], along with a compactness-type result (the $\mathcal{P}$ here refers to a subgroup being describable using equations, countable unions or intersections, and the like):

Theorem 12. If X is a Peano continuum there does not exist a strictly increasing infinite sequence of $\mathcal{P}$ normal subgroups $(G_n)_{n\in\mathbb{N}}$ of $\pi_1(X)$ such that $\bigcup_{n\in\mathbb{N}} G_n = \pi_1(X)$.

The paper [10] also has a theorem which essentially states that a free decomposition of the fundamental group of a separable, completely metrizable (i.e. *Polish*) locally path connected space cannot be non-constructive:

Theorem 13. Suppose X is locally path connected Polish and $\pi_1(X) \simeq *_{i\in I} G_i$ with each G_i nontrivial. The following hold:

(1) $\mathrm{card}(I) \leq \aleph_0$
(2) Each retraction map $r_j : *_{i\in I} G_i \to G_j$ has analytic kernel.
(3) Each G_j is of cardinality $\leq \aleph_0$ or $2^{\aleph_0}$.
(4) The map $*_{i\in I} G_i \to \bigoplus_{i\in I} G_i$ has analytic kernel.

This result about free decompositions stands in stark contrast to the case of direct sum decompositions. There are easy examples of locally path connected Polish spaces whose fundamental group has a non-constructive decomposition as a direct sum of two subgroups. Similar techniques give information about the structure of first homology, as for example (see [8]):

Theorem 14. Let X be a Peano continuum. If $\mathrm{card}(\mathrm{Torfree}(H_1(X))) < 2^{\aleph_0}$ then $H_1(X)$ is a direct sum of cyclic groups and $\mathrm{Tor}(H_1(X))$ is of bounded exponent.

I'll mention a result with W. Hojka in the vein of wild topology [21]. Recall that for a pointed space (X, x) the *reduced suspension* $\Sigma(X, x)$ is the quotient space of $X \times [-1, 1]$ where $(X \times \partial[-1, 1]) \cup (\{x\} \times [-1, 1])$ is collapsed to a single point.

Theorem 15. If (X, x) and (Y, y) are Hausdorff spaces which are first countable at their distinguished points and are totally path disconnected then the following are equivalent:

(1) $\pi_1(\Sigma(X, x)) \simeq \pi_1(\Sigma(Y, y))$
(2) There exists a bijection $f : (X, x) \to (Y, y)$ with f continuous at x and f^{-1} continuous at $y = f(x)$.

The most fascinating result of mine in exotic fundamental groups is the following answer to a conjecture of Greg Conner and James Cannon [19]:

Theorem 16. The fundamental group of the Griffiths double cone group is isomorphic to that of the harmonic archipelago.

The proof uses infinitary word combinatorics and many invocations of the axiom of choice. For a picture of these two spaces (which are subspaces of $\mathbb{R}^3$) the reader may use the hyperlink to the preprint on arXiv given in the references below.

References

1. L. Babai, *On the abstract group of automorphisms*, in Combinatorics (Temperley, editor), London Mathematical Society Lecture Note Series 52 (1981), 1-40.
2. G. Bergman, *Generating infinite symmetric groups*, Bull. London Math. Soc. 38 (2006), 429-440.
3. O. Bogopolski, S. Corson, *Abstract homomorphisms from some topological groups to acylindrically hyperbolic groups*, to appear in Math. Ann. arXiv:1907.10166
4. G. Conner, S. Corson, *On the first homology of Peano Continua*, Fund. Math. 232 (2016), 41-48. arXiv:1509.07055
5. G. Conner, S. Corson, *A note on automatic continuity*, Proc. Amer. Math. Soc. 147 (2019), 1255-1268. arXiv:1710.04787
6. Y. de Cornulier, *Strongly bounded groups and infinite powers of groups*, Comm. in Alg. 34 (2006), 2337-2347.
7. S. Corson, *Applications of Descriptive Set Theory in Homotopy Theory*, M.S. Thesis, April 2010.
8. G. Conner, S. Corson, *On subgroups of first homology*, Fund. Math. 247 (2019), 217-228. arXiv:1610.05422

9. S. Corson, *Torsion-free word hyperbolic groups are n-slender*, Int. J. Algebra Comput. 26 (2016), 1467-1482. arXiv: 1510.04326
10. ———, *On definable subgroups of the fundamental group*, Topol. Appl. 258 (2019), 336-357. arXiv:1610.00145
11. ———, *On quasi-isometry and choice*, arXiv:1609.01353
12. ———, *Automatic continuity of $\aleph_1$-free groups*, Israel J. Math 237 (2020), 267-285. arXiv: 1808.00272
13. ———, *The number of homomorphisms from the Hawaiian earring group*, J. Algebra 523 (2019) 34-52. arXiv:1801.09505
14. ———, *Freely indecomposable almost free groups with free abelianization*, J. Group Theory 23 (2020), 531-543. arXiv:1903.03334
15. ———, *The independence of Stone's Theorem from the Boolean Prime Ideal Theorem*, Proc. Amer. Math. Soc. 148 (2020), 5381-5386. arXiv:2001.06513
16. ———, *Strongly bounded locally indicable groups*, to appear in Bull. Aust. Math. Soc. arXiv:2004.13798
17. ———, *Bi-orders do not arise from total orders*, to appear in Canad. Math. Bull. arXiv:2004.13798
18. ———, *A widely connected topological space made from diamond*, arXiv:2007.10195
19. ———, *The Griffiths double cone group is isomorphic to the triple*, arXiv:2012.06794
20. ———, *Jónsson groups of various cardinalities*, to appear in Proc. Amer. Math. Soc. arXiv:2101.11035
21. S. Corson, W. Hojka, *The fundamental group of reduced suspensions*, arXiv:1712.08284
22. S. Corson, I. Kazachkov, *On preservation of automatic continuity*, Monastsh. Math. 191 (2020), 37-52. arXiv: 1901.09279
23. S. Corson, S. Shelah, *Deeply concatenable subgroups might never be free*, J. Math. Soc. Japan 71 (2019), 1123-1136. arXiv:1804.05538
24. S. Corson, S. Shelah, *Strongly bounded groups of various cardinalities*, Proc. Amer. Math. Soc. 148 (2020), 5045-5057. arXiv:1906.10481
25. S. Corson, O. Varghese, *A Nunke type classification in the locally compact setting*, J. Algebra 563 (2020), 49-52. arXiv:1912.11867
26. R. Dudley, *Continuity of homomorphisms*, Duke Math. J. 28 (1961), 587-594.
27. K. Eda, *Free σ-products and noncommutatively slender groups*, J. Algebra 148 (1992), 243-263.
28. P. Eklof, A. Mekler, *On constructing indecomposable groups in L*, J. Algebra 49 (1977), 96-103.
29. B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. 8 (1998), 810-840.
30. L. Fuchs, Infinite Abelian Groups, Vols 1,2, Academic Press, San Diego 1970, 1973.
31. C. Good, I. Tree, W. Watson, *On Stone's theorem and the axiom of choice*, Proc. Amer. Math. Soc. 126 (1998), 1211-1218.
32. M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. 53 (1981), 53-73.
33. M. Gromov, Hyperbolic Groups, Essays in Group Theory, Math. Sci. Res. Inst. Publ., 8, Springer (1987), 75-263.
34. O. Kolman, S. Shelah, *Infinitary axiomatizablity of slender and cotorsion-free groups*, Bull Belg. Math. Soc. Simon Stevin 7 (1999), 225-230
35. R. Nunke, *Slender groups*, Bull. Amer. Math. Soc. 67 (1961), 274–275.
36. A. Ol'shanskii, D. Osin, M. Sapir, *Lacunary hyperbolic groups*, Geom. Topol. 4 (2009), 2051-2140.
37. D. Osin, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016), 851-888.
38. J. Pawlikowski, *The Fundamental Group of a Compact Metric Space*, Proc. of the Amer. Math. Soc., Vol. 126, number 10, (1998) 3083-3087.
39. S. Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel J. Math. 18 (1974), 243-256.
40. S. Shelah, *On a problem of Kurosh, Jónsson groups, and applications*, Word Problems II. North-Holland Publ. Company (1980), 373-394.
41. S. Shelah, *Can The Fundamental (Homotopy) Group of a Space be the Rationals?*, Proc. Amer. Math. Soc., 103 (1988), 627–632.
42. J. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. Differential Geometry, 2 (1968), 421-446.